ASYMPTOTIC INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS DEPENDING ON ONE PARAMETER

(ASIMPTOTICHESKOE INTEGRIROVANIE DIFFERENTSIAL'NYKH URAVNENII V CHASTNYKH PROIZVODNYKH S KRAEVYMI USLOVIIAMI, ZAVISIASHCHIMI OT PARAMETRA)

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This article considers certain classical linear problems of mathematical physics: the boundary value problem for the elliptic equation, Cauchy's problem for the equation of the hyperbolic type, and the problem on the construction of the integral for an equation of arbitrary type. It is assumed that the boundary values of the solution function and its derivatives, or the arbitrary term of the equation, are rapidly oscillating functions, so that the solution of the problem depends on an arbitrarily large parameter which determines the frequency of the oscillations.

The solution is based on the use of the method proposed in monograph [1]. The author presented the statement of the problem and parts of the results at the Third All-Union Mathematical Congress. A closely-related problem was treated in a note by Vishik and Liusternik [2]. In this a different method was used, one developed by the authors in an earlier publication [3]. This article [3] contains a detailed survey of publications on the asymptotic integration of partial differential equations. This topic is gaining interest among mathematicians in the U.S.S.R. as well as abroad.

In the present article it is assumed that the number of independent variables is two, but the method used is also applicable to the case of more variables. The emphasis is here placed on the practical side of the problem, i.e. on rapid ways of finding the explicit solution.

The approximate solution of all the above problems, including those with boundary conditions, can be reduced to the repeated solution of Cauchy's problem (in a complex region) for first-order equations. In this connection it was found convenient to make no distinction between the boundary and the initial conditions, which are described in this article by the unifying term "contour conditions".

1. 1. Let the following linear differential equation of order l be given

$$L(\Phi) \equiv \sum_{\nu=0}^{\nu=l} \sum_{j=0}^{j=\nu} a_{j,\nu-j}^{(\nu)} \frac{\partial^{\nu}\Phi}{\partial\alpha^{j}\,\partial\beta^{\nu-j}} = 0 \tag{1.1}$$

where (α, β) are independent variables, Φ is the function to be determined and $a_{jk}^{(\nu)}$ are real functions of α and β.

It is assumed that the functions are sufficiently smooth to guarantee that the characteristics L (real or imaginary) will be unique, and that, within the considered region plus its boundary, the equation (1.1) has no singular points, i.e. points at which all the coefficients $a_{jk}^{(l)}$ are simultaneously zero.

2. We will seek a solution of equation (1.1) in the form

$$\Phi = e^{kf}\Phi_{*} = e^{kf} \sum_{u=0}^{u=R} k^{-u}\Phi_{u} \qquad (\Phi_0 \not\equiv 0) \tag{1.2}$$

where k is a real constant, and f and Φ_u are functions of α and β. We will call the function f the change function, and Φ_* the intensity function. The terms $\Phi_u (u = 1, \ldots, R-1)$ will be called the coefficients of the expansion of the intensity function, Φ_R will stand for the remainder. We impose the following additional requirement: the change function and the coefficients of the expansion of the intensity function are independent of k.

3. Let $D^{(s,t)}$ be the symbol indicating the s-th partial derivative with respect to α, and the t-th partial derivative with respect to β. Then

$$D^{(s,t)}(e^{kf}\Phi_*) = e^{kf}k^{s+t}\left(f_\alpha + k^{-1}\frac{\partial}{\partial\alpha}\right)^{s}\left(f_\beta + k^{-1}\frac{\partial}{\partial\beta}\right)^{t}\Phi_* \tag{1.3}$$

(f_α, f_β are the partial derivatives of f with respect to α and β respectively).

The symbolic product standing on the right-hand side of the last equation can be expanded in descending powers of k. We thus obtain

$$e^{-kf}D^{(s,t)}(e^{kf}\Phi_*) = k^{s+t}\left\{\sum_{u=0}^{u=s+t} k^{-u}D_u^{(s,t)}\right\}\Phi_* \tag{1.4}$$

where $D_u^{(s,t)}$ is some differential operator of order u. In particular

$$D_0^{(s,t)} = f_\alpha^s f_\beta^t \tag{1.5}$$

$$D_1^{(s,t)} = \frac{\partial}{\partial f_\alpha}\{D_0^{(s,t)}\}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial f_\beta}\{D_0^{(s,t)}\}\frac{\partial}{\partial\beta} + \frac{1}{2}\frac{\partial^2}{\partial f_\alpha{}^2}\{D_0^{(s,t)}\}\frac{\partial^2 f}{\partial\alpha^2} +$$

$$+ \frac{\partial^2}{\partial f_\alpha \partial f_\beta}\{D_0^{(s,t)}\}\frac{\partial^2 f}{\partial\alpha\,\partial\beta} + \frac{1}{2}\frac{\partial^2}{\partial f_\beta{}^2}\{D_0^{(s,t)}\}\frac{\partial^2 f}{\partial\beta^2}$$

With the aid of equations (1.4) and (1.1) one can write

$$L(e^{kf}\Phi_*) = e^{kf}k^l \sum_{v=0}^{v=l} k^{-v}L_v(\Phi_*) \tag{1.6}$$

where

$$L_0 = \sum_{j=0}^{j=l} a_{j,\,l-j}^{(l)} f_\alpha{}^j f_\beta{}^{l-j} \tag{1.7}$$

$$L_1 = \frac{\partial}{\partial f_\alpha}\{L_0\}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial f_\beta}\{L_0\}\frac{\partial}{\partial\beta} + \frac{1}{2}\frac{\partial^2}{\partial f_\alpha{}^2}\{L_0\}\frac{\partial^2 f}{\partial\alpha^2} +$$

$$+ \frac{\partial^2}{\partial f_\alpha\,\partial f_\beta}\{L_0\}\frac{\partial^2 f}{\partial\alpha\,\partial\beta} + \frac{1}{2}\frac{\partial^2}{\partial f_\beta{}^2}\{L_0\}\frac{\partial^2 f}{\partial\beta^2} + \sum_{j=0}^{j=l-1} a_{j,\,l-1-j}^{(l-1)} f_\alpha{}^j f_\beta{}^{l-1-j} \tag{1.8}$$

and $L\gamma$ is a linear differential operator of order γ whose coefficients are polynomials in the derivatives of f, and are linear functions of $a_{jk}{}^{(\nu)}$.

4. Substituting the expression (1.2) into (1.6), and expanding the result in descending powers of k, we obtain

$$L(\Phi) \equiv e^{kf}\left\{k^l \sum_{v=0}^{v=l} k^{-v} \sum_{u=0}^{u=R} k^{-u}L_v(\Phi_u)\right\} = 0 \tag{1.9}$$

Dropping the exponential factor and carrying out some obvious transformations we obtain*:

$$k^l \sum_{r=0}^{r=R+l} \sum_{u=0}^{u=r} k^{-r}L_{r-u}(\Phi_u) = 0 \qquad (r-u \leqslant l,\ u \leqslant R)$$

* Here and in the sequel the indicated sums contain only those terms whose summation indices satisfy the auxiliary inequalities indicated in the parentheses.

Let us require that all the coefficients of the powers of k from l to $l-R$ be zero in the left-hand side of this last equation. This equation then reduces to the system

$$\sum_{u=0}^{u=r} L_{r-u}(\Phi_u) = 0 \qquad (r-u \leqslant l;\ u \leqslant R;\ r=0,1,\ldots,R) \tag{1.10}$$

$$\sum_{r=R+1}^{r=R+l} \sum_{u=0}^{u=r} k^{-r} L_{r-u}(\Phi_u) = 0 \qquad (r-u \leqslant l,\ u \leqslant R) \tag{1.11}$$

5. It is easy to prove that the relations (1.10) lead to a recurrence system of equations for the determination of the change function and of the coefficients of the expansion of the intensity function.

Putting $r = 0$ in (1.10) and taking into account equation (1.7), we obtain for the determination of the change function the following first-order differential equation of degree l:

$$L_0 \equiv \sum_{j=0}^{j=l} a_{j,\,l-j}^{(l)} f_\alpha{}^j f_\beta{}^{l-j} = 0 \tag{1.12}$$

Making use of (1.12) we can transform (1.10) to the form

$$L_1(\Phi_0)=0, \quad L_1(\Phi_{r-1}) = -\sum_{u=0}^{u=r-2} L_{r-u}(\Phi_u) \quad (r-u \leqslant l;\ u \leqslant R;\ r=2,\ldots,R) \tag{1.13}$$

By means of these equations it is possible to determine all the coefficients of the expansion of the change function.

6. To determine the remainder term, we have the equation (1.11). Dividing out the factor k^{-R-1} in this equation and changing the summation index by means of the formula $r - R = \rho + 1$, we can rewrite equation (1.11) as

$$\sum_{\rho=0}^{\rho=l-1} k^{-\rho} L_{\rho+1}(\Phi_R) = -\sum_{\rho=0}^{\rho=l-1} \sum_{u=0}^{u=R-1} k^{-\rho} L_{R+\rho+1-u}(\Phi_u) \quad (R+\rho+1-u \leqslant l) \tag{1.14}$$

7. We have found a recurrence process for the successive determination of the function f, Φ_0, Φ_1, ..., Φ_{R-1}, Φ_R. Here each successive function can be obtained from the known preceding functions by solving one differential equation. To determine the change function, we have a first-order differential equation of degree l: for the determination of each of the coefficients of the expansion of the intensity function, we have a first-order linear differential equation. The remainder term satisfies a linear differential equation of order l (with the same principal part as the original equation).

8. The equation (1.12), which determines the change function, is the differential equation of the characteristics of the operator L. Therefore, the solutions of equation (1.12) will be such functions, and only such, as are constant on an arbitrary curve of some family of characteristics of L. Hence, every integral of type (1.2) can be put in correspondence with some family of characteristics of L, namely, with a family on whose curves the change function preserves its constant value. Integrals which correspond to different families of characteristics of L, will be said to be essentially different integrals. In the case considered (when the characteristics are unique) there exist essentially different families of integrals, each of which corresponds to its own family of characteristics.

The problem of solving the nonlinear equation (1.12) is easily reduced to the problem of integrating first-order linear equations. Indeed, in equation (1.12), L_0 represents a homogeneous polynomial of degree l in f_α and f_β. Therefore, this expression can be represented as a product of l real or complex factors

$$L_0 = \prod_{\tau=1}^{\tau=l} (A_{\tau 1} f_\alpha + A_{\tau 2} f_\beta)$$

and hence, each solution of equation (1.12) must satisfy at least one of the equations

$$A_{\tau 1} f_\alpha + A_{\tau 2} f_\beta = 0 \qquad (\tau = 1, \ldots, l) \tag{1.15}$$

Arbitrarily selecting one of equations (1.15) for the construction of f, we thereby also select that family of characteristics to which the solution integral corresponds.

9. We shall consider the question regarding the singular points of equations (1.12), (1.13) and (1.14). The equation (1.12) has no singular points, for the $a_{j,l-j}^{(l)}$ are, by hypothesis, sufficiently smooth and cannot vanish simultaneously.

In the sequel it will always be assumed that the solutions of equations (1.12) and (1.13) are sufficiently smooth within the regions that interest us. Therefore, there can occur singular points in equations (1.13) and (1.14) only when the coefficients of the highest derivatives of the solution function vanish simultaneously. For equation (1.14) this cannot happen, for if it did, then, contrary to hypothesis, equation (1.1) would also have singular points. In equations (1.13), the coefficients of the highest derivatives can vanish simultaneously only when the following equations are satisfied simultaneously with (1.12):

$$\frac{\partial}{\partial f_\alpha}\{L_0\} = 0, \qquad \frac{\partial}{\partial f_\beta}\{L_0\} = 0$$

These equations can hold only under the following conditions: (a) at stationary points of the function R, where $f_a = f_\beta = 0$ (if $l > 1$); (b) at points where the characteristics of the family (corresponding to the given integral) are tangent to curves of some other family of characteristics.

2. In this and the following sections we will show by concrete examples that the method described in the preceding section makes it possible to construct integrals sufficiently general to yield solutions of some classical problems in the theory of differential equations.

1. We will consider a simply connected region $\bar{\Gamma} = \Gamma + \gamma$ bounded by the contour γ. Let the parameters (a, β) be the coordinates of a system similar to that of polar coordinates, i.e. the contour γ is given by the equation $a = a_0 > 0$, the region Γ is determined by the inequalities $0 \leqslant a \leqslant a_0$, and $0 \leqslant \beta \leqslant 2\pi$, while the correspondence between the points of the region and the pairs (a, β) is a reciprocal one-to-one correspondence except for the point $a = 0$ and the lines $\beta = 0$, $\beta = 2\pi$.

2. We will assume that the operator L is elliptic everywhere in Γ (hence, l is even). We set ourselves the following problem A. It is required to construct within Γ a solution of equation (1.1) satisfying the following boundary conditions on γ:

$$\frac{\partial^\mu \Phi}{\partial \alpha^\mu} \equiv D^{(\mu,0)}(\Phi) = k^\mu g^{(\mu)} e^{ik\varphi} \qquad (\mu = 0, 1, 2, \ldots, {}^1/_2\, l - 1) \tag{2.1}$$

where $g^{(\mu)}$ and ϕ are given functions of β which do not depend on k.

The parameters of the problem are assumed to be sufficiently smooth, i.e. γ has a smoothness of high enough order, while $g^{(\mu)}$ and $e^{ik\phi}$ are sufficiently smooth not only as functions of β, but also as functions of a point on the contour γ.

The function ϕ will be considered to be a real, monotonic increasing (decreasing) function of β. The function $\phi'(\beta)$ will therefore always be positive (negative); the functions $g^{(\mu)}$ may be complex-valued.

If $g^{(\mu)} = \text{const}$, $\phi = \beta$, and k is an integer, then the problem A reduces to the classical problem of the theory of elliptic differential equations, when the boundary functions are expanded into complex Fourier series and when in each there is retained only one term of sufficiently high order of k.

3. The solution of problem A will be constructed as the sum of integrals of type (1.2), which correspond to certain families of characteristics of the operator L.

By $f^{(q)}$, $\Phi_u^{(q)}$, let us denote the component integrals which correspond to the q-th family of the characterics of the operator L and let us ask whether it is possible to subject $f^{(q)}$ to the condition

$$f^{(q)} = i\varphi(\beta), \qquad \operatorname{Re}\{f_\alpha^{(q)}\} > 0 \qquad \text{on } \gamma \tag{2.2}$$

Obviously, the first of the conditions (2.2) can be fulfilled for any family of characteristics by solving Cauchy's problem for the corresponding equation (1.15). It remains for us to consider the second conditions in (2.2). Because of the ellipticity of L, all families of its characteristics are imaginary, and hence, all $f^{(q)}$ are complex functions. Furthermore, to every $f^{(s)}$ there corresponds an $f^{(t)} = \pm \bar{f}^{(s)}$ (the bar above a quantity indicates the complex conjugate).

From condition (2.2) it follows that $\gamma f_\beta{}^{(q)} = i\phi'(\beta)$, that is, $f_\beta{}^{(q)}$ takes on purely imaginary values at every point of the contour γ. But, for the elliptic operator L, the coefficients (A_{1r}, A_{2r}) in equations (1.15) are complex, and none of these equations can be satisfied if $f_\alpha{}^{(q)}$ and $f_\beta{}^{(q)}$ are purely imaginary. Hence, the real part of $f_\alpha{}^{(q)}$ must be different from zero at each point of γ. Thus, if $f^{(q)}$ is subjected to the first condition of (2.2), then the sign of the real part of $f_\alpha{}^{(q)}$ on γ is uniquely determined. Moreover, it is easy to verify that if $f^{(s)}$ and $f^{(t)}$ and $\bar{f}^{(t)} = \pm \bar{f}^{(s)}$ are subjected to the same condition (2.2), then the signs of the real parts of $f_\alpha{}^{(s)}$ and $f_\alpha{}^{(t)}$ will differ on γ. From this it follows that there exist exactly $l/2$ families of characteristics of the operator L for which both the (2.2) conditions can be satisfied.

4. Let us enumerate the families of the characteristics of L in such a way that in the first places there occur the families for which both (2.2) conditions are satisfied. We are seeking a solution of the problem A in the form

$$\Phi = \sum_{q=1}^{q=\frac{1}{2}l} \left(e^{kf^{(q)}} \sum_{u=0}^{u=R} k^{-u}\Phi_u{}^{(q)}\right)$$

assuming that all $f^{(q)}$ satisfy the conditions (2.2). Interchanging $D^{(\mu,0)}$ for L in (1.9), we obtain

$$D^{(\mu,0)}(\Phi) = \sum_{q=1}^{q=\frac{1}{2}l} e^{kf^{(q)}} \left(k^\mu \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R} k^{-u} D_v{}^{(\mu,0)}(\Phi_u{}^{(q)})\right)$$

$$(\mu = 0, 1, 2, \ldots, \tfrac{1}{2}l - 1)$$

Substituting this result into (2.1), and taking into consideration that the $f^{(q)}$ satisfy conditions (2.2), we find that

$$\sum_{q=1}^{q=\frac{1}{2}l} k^\mu \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R} k^{-u} D_{v,q}{}^{(\mu,0)}(\Phi_u{}^{(q)}) = k^\mu g^{(\mu)} \qquad \text{on } \gamma \tag{2.3}$$

$$(\mu = 0, 1, 2, \ldots, \tfrac{1}{2}l - 1)$$

Let us in this expression require the coefficients of corresponding powers (from μ to $\mu - R + 1$) of k on the left- and right-hand sides of the equations to be equal. After some transformations, this yields

$$\sum_{q=1}^{q=\frac{1}{2} l} D_{0,q}^{(\mu,0)}(\Phi_0^{(q)}) = g^{(\mu)} \qquad \text{on } \gamma \tag{2.4}$$

$$\sum_{q=1}^{q=\frac{1}{2} l} D_{0,q}^{(\mu,0)} \quad (\Phi_r{}^{(q)}) = - \sum_{q=1}^{q=\frac{1}{2} l} \sum_{u=0}^{u=r-1} D_{r-u,q}^{(\mu,0)}(\Phi_u{}^{(q)}) \qquad \binom{r-u\leqslant\mu}{r=1,\ldots,R-1} \text{ on } \gamma \tag{2.5}$$

$$\sum_{q=1}^{q=\frac{1}{2} l} \sum_{\rho=0}^{\rho=\mu} k^{-\rho} D_{\rho,q}^{(\mu,0)} \quad (\Phi_R^{(q)}) = - \sum_{q=1}^{q=\frac{1}{2} l} \sum_{\rho=0}^{\rho=\mu-1} \sum_{u=0}^{u=R-1} k^{-\rho} D_{R+\rho-u,q}^{(\mu,0)}(\Phi_u{}^{(q)}) \qquad \text{on } \gamma$$

$$(R+\rho-u\leqslant\mu) \tag{2.6}$$

5. The expression $D_{0,q}^{(\mu,0)}$ can be given with the help of equation (1.5) as

$$D_{0,q}^{(\mu,0)} = (f_\alpha{}^{(q)})^\mu$$

From this it follows that (2.4) represents a system of $l/2$ linear algebraic equations in the $\Phi_0{}^{(q)}$ ($q = 1, 2, \ldots, l2$).

If the functions $\Phi_0{}^{(q)}, \Phi_1{}^{(q)}, \ldots, \Phi_{r-1}^{(q)}$ are known, and one can therefore construct the contour values of these functions and the required number of their derivatives, then the right-hand sides of (2.5) are known functions and these relations constitute a system of linear algebraic equations in the contour values $\Phi_r{}^{(q)}$ ($q = 1, \ldots, l/2$).

All the systems mentioned have the same determinant:

$$\Delta = \| (f_\alpha{}^{(q)})^\mu \| \qquad (q = 1, 2, \ldots, \tfrac{1}{2} l, \ \mu = 0, 1, \ldots, \tfrac{1}{2} l - 1)$$

This is the Vandermond determinant, which can be equal to zero only on condition that at least two functions f_a with two different superscripts are equal to each other.

This can occur only at those points on γ where two characteristics of L, which belong to different families, are tangent to each other*, for all the $f_\beta{}^{(q)}$ equal to each other on γ and for arbitrary β. This is a

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* Here and in the sequel it will be assumed that the real or imaginary curves ϕ = const and ψ = const are tangents to each other at every point where (ϕ_a, ϕ_β) are proportional to (ψ_a, ψ_β).

consequence of the conditions (2.2).

We will exclude from consideration the case when γ contains such points (they are also singular points of the equations which determine the coefficients of the expansion of the intensity function). The contour conditions (2.4) and (2.5) can now be changed to the form:

$$\Phi_s^{(q)} = \overline{\Phi}_s^{(q)} \qquad (s = 0, 1, \ldots, R-1) \quad \text{on } \gamma \tag{2.7}$$

where the $\overline{\Phi}_s^{(q)}$ are known as soon as the functions $\Phi_0^{(q)}$, $\Phi_1^{(q)}$, ..., $\Phi_{s-1}^{(q)}$ are determined in a neighborhood of γ.

The relation (2.6) gives the boundary conditions for the remainder terms $\Phi_R^{(q)}$. Here we have $l/2$ boundary conditions imposed upon $l/2$ functions $\Phi_R^{(q)}$. Each of the $\Phi_R^{(q)}$ functions satisfies an equation of order l, and the problem of constructing the remainder terms remains indeterminate (as a Cauchy problem). We will return to this problem later.

6. It has thus been shown that it is possible to construct a solution of problem A in the neighborhood of γ; in particular, this solution satisfies the second of the (2.2) conditions. Henceforth we will refer to this condition as the condition of damping. We have found contour conditions which must be taken into consideration when constructing the change function and the coefficients of the expansion of the intensity function. Moreover, we have found that the construction of each of the functions enumerated can be reduced to the solution of Cauchy's problem for a first-order linear equation. Finally, we have derived contour conditions (insufficient in number) to determine the remainder terms.

3. We will next show that under certain known conditions, the solution of problem A (constructed in the preceding section in a neighborhood of γ) can be extended over the entire region Γ, and that the problem on the construction of the remainder term can be supplemented to the extent that $\Phi_R^{(q)}$ remains bounded for arbitrarily large k.

1. We introduce into consideration a region Γ_ϵ contained between γ (the line $a = a_0$) and γ_ϵ (the line $a = a_0 - \epsilon$), where ϵ is a positive, sufficiently small number. Moreover, we assume that (a) equation (1.13) with contour conditions (2.7) has (in Γ_ϵ) sufficiently smooth solutions Φ_0, Φ_1, ..., Φ_{R-1}; (b) equations (1.14) with contour conditions (2.6) can be made determinate by the addition of a certain number of auxiliary conditions under which these equations (1.14) will (in Γ_ϵ) have sufficiently smooth solutions; (c) equation

$$L(\Phi) = \Psi \tag{3.1}$$

with homogeneous contour conditions of type (2.1), in the entire region Γ has a sufficiently smooth solution for an arbitrary sufficiently smooth function Ψ.

Conditions under which statements (a) and (c) are true have been treated in the literature, and we will not dwell on them. We will, however, return to a consideration of the question of under what conditions statement (b) may hold.

2. In consequence of (2.2), the real part of f_α will be non-positive* in Γ_ϵ for sufficiently small ϵ. Owing to assumptions (a) and (b), the functions $\Phi_u^{(q)}$ ($u = 0, 1, \ldots, R$) are bounded. Therefore, on an arbitrary contour γ ($0 < \eta < \epsilon$), the absolute value of the function

$$\Phi^{(q)} = e^{kf^{(q)}} \sum_{u=0}^{u=R} k^{-u}\, \Phi_u^{(q)} \tag{3.2}$$

will be of order $0(k^{-\nu})$, where ν is an arbitrary positive number. A sufficient number of derivatives of $\Phi^{(q)}$ will have the same property.

As in an earlier article [3], we introduce a smoothing function ψ, which has the value one in the region Γ_η and value zero in $\Gamma - \Gamma_\epsilon$, and has derivatives of all orders at every point of Γ, and we consider the function

$$\Phi_*^{(q)} = \psi\Phi^{(q)} + \Phi^0 \tag{3.3}$$

where $\Phi^{(q)}$ is one of the $l/2$ solutions of (1.1) constructed in the neighborhood of γ by the method of the preceding section, while Φ^0 is a function which, everywhere in Γ, satisfied the equation

$$L(\Phi^0) = -L\ (\psi\Phi^{(q)})$$

This implies that $\Phi^{(q)}$ is a solution of (1.1).

On the right-hand side of the equation last displayed there is a function, whose value is zero in Γ_η and in $\Gamma - \Gamma_\epsilon$, for $\psi = 1$ in Γ_η and $\Phi^{(q)}$ satisfies the equation $L(\Phi) = 0$ in the neighborhood of γ, while in $\Gamma - \Gamma_\epsilon$, the function $\psi = 0$. In $\Gamma_\epsilon - \Gamma_\eta$, the function ψ and a sufficient number of its derivatives are bounded, and the function $\Phi^{(q)}$ is of type $0(k^{-\nu})$. Thus we have an equation of form (3.1), which, in consequence of hypothesis (c), has a solution of the type $0(k^{-\nu})$. This means that with the aid of formula (3.3), each of the functions $\Phi^{(q)}$, which in Section 2 were constructed for neighborhoods of γ only, has now been extended over the region Γ. Furthermore, all boundary conditions on γ are preserved, and the extended function $\Phi^{(q)}$ satisfies the equation (1.1).

3. We now return to the problem of making the construction of the

* This follows from a theorem on uniform continuity.

remainder term determinate. The equation (1.14), which has to be satisfied by the function Φ_R, is of order l with a small parameter k^{-1}, which appears as a coefficient of the highest-order derivatives. In the paper by Vishik and Liusternik cited above [3], an equation of this type was considered from a general viewpoint. In this paper the authors introduce the concepts of the "limiting" (degenerate) problem and of the regularity of the degeneration. By the limiting problem is meant the problem of solving the degenerated equation which results if $k^{-1} = 0$ and when one has the appropriate number of contour conditions (only a part of the contour conditions originally given will remain, if the limiting problem is correctly formulated). The degeneration of the original problem to the limiting problem is said to be a regular degeneration if the solution of the limiting problem uniformly converges, as $k \to \infty$, to the solution of the original problem at every point that is not on γ. In the neighborhood of γ there takes place a rapid convergence to zero of the absolute value of the additional terms, which permit a compensation for the mismatch in the contour conditions; (in the previous paper [3] these additional terms were called boundary layer terms, while in monograph [1] they were referred to as integrals with a given support contour).

Vishik and Liusternik derived the condition for the degeneration to be regular. It consists of the requirement that the so-called auxiliary characteristic equation shall have as many roots with positive real parts (when motion into the region corresponds to a decrease in α) as the number of boundary conditions that had to be dropped in the passage from the original problem to the limiting problem.

It is not difficult to discover, by following the arguments presented in reference [3], that if the conditions for the regularity of degeneration are not satisfied, then the additional terms described above can still be constructed, but they will no longer decrease (with an increase in k) as one passes from the boundary to the interior of the region. If, however, the solutions of the limiting problem satisfy all boundary conditions of the original problem, then the additional terms will not appear.

4. Let us next verify whether the condition for the regularity of the degeneration is satisfied in the case that interests us. If the boundary conditions are given for the boundary $\alpha = \alpha_0$, then the auxiliary characteristic equation for the operator

$$\sum_{\rho=0}^{\rho=l-1} k^{-\rho} L_{\rho+1} \tag{3.4}$$

which appears on the left-hand side of the equation (1.14), is constructed in the following way: from each of the differential operators $L_{\rho+1}$ is taken only the term that contains the derivative with respect to α of order $\rho + 1$, and this derivative is replaced by λ, where λ is an unknown (all functions are hereby replaced by their contour values). One can show that these operations yield the following equation

$$[a_{l0}^{(l)} (f_\alpha + k^{-1}\lambda)^l_{\alpha=\alpha_0} = 0 \tag{3.5}$$

all of whose roots are given by the formula

$$\lambda = - k \{f_\alpha\}_{\alpha=\alpha_0}$$

From this and from conditions (2.2) it follows that the auxiliary characteristic equation has no root with a positive real part. This means that none of the boundary conditions which might make the problem on the construction of the remainder term Φ_R determinate can be given arbitrarily. These boundary conditions must be so designed as to be satisfied by the solution of the limiting problem.

5. We will next make a more detailed study of the Cauchy problem to which the determination of the remainder terms in Γ_ϵ can be reduced. We have $l/2$ equations of form (1.14) for each of the $\Phi_R^{(q)}$ and $l/2$ boundary conditions of form (2.6). We construct the limiting (degenerate) equation for equation (1.14)

$$L_1 (\Phi_R^{(q)}) = - \sum_{u=0}^{u=R-1} L_{R+1-u} (\Phi_u^{(q)}) \qquad (R + 1 - u \leqslant l) \tag{3.6}$$

and attach thereto the boundary conditions

$$\Phi_R^{(q)} \big|_{\alpha=\alpha_0} = \Psi_0^{(q)} \tag{3.7}$$

where $\Psi_0^{(q)}$ are some still undetermined functions on the contour γ.

Let us suppose that the limiting Cauchy problem (3.6) and (3.7) has been solved. Then one can find $\Psi_1^{(q)}$, $\Psi_2^{(q)}$, ..., $\Psi_{l-1}^{(q)}$, which are the contour values of the derivatives of Φ_R with respect to α, of the respective order. All of these can be expressed in terms of the $\Psi_0^{(q)}$ and their derivatives with respect to β. Substituting these results into boundary conditions (2.6), we obtain, for the $\Psi_0^{(q)}$, a system of $l/2$ ordinary differential equations (with β as the independent variable).

Without taking time to investigate this system, we suppose it to possess sufficiently smooth periodic solutions. Then the problem of the $\Phi_R^{(q)}$ functions will be properly defined (as a Cauchy problem) in the following way: for each of the equations of type (1.14) there are given contour values of the function $\Phi_R^{(q)}$ and of its $l - 1$ derivatives; the solution of the limiting problem will satisfy all these contour conditions, while contour conditions (2.6) are satisfied automatically.

For the problem on the construction of the remainder terms $\Phi_R^{(q)}$, which has thus been made determinate, hypothesis (*b*) will be satisfied.

6. Thus, within the framework of the above hypotheses, the solution of problem *A* will decrease rapidly, for large enough *k*, as one moves from the contour to the interior of the region; this solution can be approximately constructed (by neglecting the remainder terms) to any degree of accuracy by the successive solution of Cauchy's problem for first-order linear differential equations.

4. We will consider another classical problem of mathematical physics.

1. Let L be a completely hyperbolic operator of order l, which possesses l different families of real characteristics. The problem *B*, which we will consider, consists of the construction of a solution of equation (1.1) which on some contour γ, not tangent to any one of the characteristics of L, satisfies the conditions

$$D^{(\mu,0)}(\Phi) = k^{\mu} g^{(\mu)} e^{ik\varphi} \quad (\mu = 0, 1, \ldots, l-1) \text{ on } \gamma \tag{4.1}$$

where $D^{(\mu,0)}$ is a symbol that stands for the μ-th order derivative in the normal direction to γ; $g^{(\mu)}$ and ϕ do not depend on k and are sufficiently smooth functions defined on γ. Moreover, ϕ is a real function, while the $g^{(\mu)}$ are complex. Such boundary conditions will be met, for example, when we try to solve the classical Cauchy problem by expanding the boundary functions into complex Fourier series, and then concentrate our attention on a single term of high enough order.

For the sake of simplicity, let us suppose that the contour γ coincides with the line $a = a_0$, which, obviously, does not restrict the generality of the results. Then $D^{(\mu,0)}$ becomes a symbol for the l-th derivative with respect to a, as it was in Section 1.*

2. The solution of problem *B* may be sought as a sum of integrals of form (1.2) corresponding to all the families of characteristics of the

* When this article was already in the press, it became known to the author that in the work of Lax [4] problem *B* is considered in much greater detail than here. Lax solves problem *B* for first-order hyperbolic systems of equations in an arbitrary number of independent variables. He uses a very similar method to that used here and in monograph [1].

operator L:

$$\Phi = \sum_{q=1}^{q=l} \left\{ e^{kf^{(q)}} \sum_{u=0}^{u=R} k^{-u} \Phi_u{}^{(q)} \right\}$$

We impose the following boundary conditions on all change functions

$$f^{(q)} = i\varphi(\beta) \quad \text{on } \gamma \tag{4.2}$$

For a completely hyperbolic operator in equation (1.15), the coefficients (A_{1r}, A_{2r}) are proportional to a pair of real functions. Therefore, in consequence of (4.2), the functions $f^{(q)}$ will be pure imaginary. The contour conditions (4.1) can be represented, with the aid of (4.2), in the form

$$\sum_{q=1}^{q=l} k^{\mu} \sum_{v=0}^{v=\mu} k^{-v} \sum_{u=0}^{u=R} k^{-u} D_{v,q}^{(\mu,0)} (\Phi_u{}^{(q)}) = k^{\mu} g^{(\mu)} \quad \text{on } \gamma \tag{4.3}$$

which is entirely analogous to the relation (2.3). The remaining arguments can be carried out along the same line as that used in section 2. Making the requirement that in the relation (4.3) the coefficients of corresponding powers of k, from μ to $\mu - R - 1$, shall be equal on the right and left sides of the equations, we obtain (for $\mu = 0, 1, \ldots, l - 1$)

$$\sum_{q=1}^{q=l} D_{0,q}^{(\mu,0)} (\Phi_0{}^{(q)}) = g^{(\mu)} \qquad \text{on } \gamma$$

$$\sum_{q=1}^{q=l} D_{0,q}^{(\mu,0)} (\Phi_r{}^{(q)}) = -\sum_{q=1}^{q=l} \sum_{u=0}^{u=r-1} D_{r-u,q}^{(\mu,0)} (\Phi_u{}^{(q)}) \; (r - u \leqslant \mu;\ r = 1, \ldots, R-1) \quad \text{on } \gamma$$

$$\sum_{q=1}^{q=l} \sum_{\rho=0}^{\rho=\mu} k^{-\rho} D_{\rho,q}^{(\mu,0)} (\Phi_R{}^{(q)}) = -\sum_{q=1}^{q=l} \sum_{\rho=0}^{\rho=\mu} \sum_{u=0}^{u=R-1} k^{-\rho} D_{R+\rho-u,q}^{(\mu,0)} (\Phi_u{}^{(q)}) \quad (R + \rho - u \leqslant \mu) \text{ on } \gamma \tag{4.4}$$

The first two of these relations make it possible to reduce the determination of the coefficients of the expansion of the intensity function to the successive solution of Cauchy's problem for first-order linear differential equations. Here, as in Section 2, it is necessary to require that the Vandermond determinant shall

$$\Delta = \| (f_\alpha^{(q)})^{\mu} \| \quad (q = 1, 2, \ldots, l;\ \mu = 0, 1, \ldots, l-1)$$

be different from zero, that is, that on the curve γ there shall be no point at which two characteristics belonging to different families are tangent to each other.

This, on the basis of (4.4), establishes l contour conditions for the remainder terms $\Phi_R^{(q)}$. Next, it is necessary to make the problem on the construction of the $\Phi_R^{(q)}$ determinate. For this purpose one can, for example, impose the requirement that this problem shall have a regular degeneration. The auxiliary characteristic equation for problem B is given by (3.2). All its roots are pure imaginary. This follows from equation (4.2). Therefore, the problem on the construction of Φ_R can be made determinate in the same way as in Section 3.

5. The application of the proposed method to the construction of a particular integral.

1. Suppose we are given the equation

$$L(\Phi) = \Psi(\alpha, \beta)\, e^{kf(\alpha, \beta)} \tag{5.1}$$

where f is a pure imaginary and sufficiently smooth function which has no singular points in the region that interests us; $\Psi(\alpha, \beta)$ is a sufficiently smooth complex function; k is a sufficiently large real constant. Let us consider the problem of finding a particular integral of (5.1) in the form

$$\Phi = k^p e^{kf(\alpha, \beta)} \sum_{u=0}^{u=R} k^{-u} \Phi_u \tag{5.2}$$

where the number p, the functions $\Phi_0, \Phi_1, \ldots, \Phi_{R-1}$ (independent of k) and the function Φ_R (depending on k) are to be determined.

2. The substitution of (5.2) into (5.1), and the division by an exponential factor, result in the following equation

$$k^{l+p} \sum_{v=0}^{v=l} k^{-v} \sum_{u=0}^{u=R} k^{-u} L_v(\Phi_u) = \Psi \tag{5.3}$$

Let us suppose that the expression

$$L_0 \equiv \sum_{j=0}^{j=l} a_{j,\, l-j}^{(l)} f_\alpha^j f_\beta^{l-j}$$

is different from zero at every point of the region under consideration; that is, the level lines of the function f are not tangent to the characteristics of L. One can then set $p = -l$ and require the coefficient of the corresponding powers of k (from 0 to $-R+1$) to be equal on the two sides of equation (5.3). We then obtain recurrence formulas to determine the coefficients of the expansion of the intensity function,

$$\Phi_0 = \frac{\Psi}{L_0}, \Phi_r = -\frac{1}{L_0} \sum_{p=1}^{p=r} L_p(\Phi_{r-p}) \qquad (p \leqslant l;\ r = 1, \ldots, R-1) \quad (5.4)$$

and the l-th order linear differential equation to determine the remainder term

$$\sum_{\rho=0}^{\rho=l} k^{-\rho} L_\rho(\Phi_R) = -\sum_{\rho=0}^{\rho=l} \sum_{u=0}^{u=R-1} k^{-\rho} L_{R+\rho-u}(\Phi_u) \qquad (R+\rho-u \leqslant l) \quad (5.5)$$

3. Next, let us suppose that in the entire region under consideration the function f satisfies the equation

$$L_0 \equiv \sum_{j=0}^{j=l} a_{j,\,l-1}^{(l)} f_\alpha^j f_\beta^{l-j} = 0$$

that is, the level lines of the function f everywhere coincide with the characteristics of L. Then equation (5.1) will take on the form

$$k^{l+p-1} \sum_{v=1}^{v=l} k^{-(v-1)} \sum_{u=0}^{u=R} k^{-u} L_v(\Phi_u) = \Psi \quad (5.6)$$

Let us put $p = -l + 1$, and again require the coefficients of corresponding powers of k (from 0 to $-R+1$) to be equal on the two sides of the equation (5.6). We obtain the following system of recurrence equations to determine the coefficients of the expansion of the intensity function

$$L_1(\Phi_0) = \Psi, \qquad L_1(\Phi_r) = -\sum_{p=2}^{\rho=r} L_p(\Phi_{r-p}) \qquad (p \leqslant l;\ r = 1, \ldots, R-1) \quad (5.7)$$

To determine the remainder term, we obtain the linear differential equation of order l

$$\sum_{\rho=1}^{\rho=l} k^{-\rho} L_\rho(\Phi_R) = -\sum_{\rho=0}^{\rho=l} \sum_{u=0}^{u=R-1} k^{-\rho} L_{R+\rho-u}(\Phi_u) \qquad (R+\rho-u \leqslant l) \quad (5.8)$$

The singular points of the differential equation (5.7) were investigated in Section 1. These points can occur only at points of tangency of characteristics belonging to different families (the case when f has stationary points is excluded from consideration).

4. Let us postulate that equations (5.5) and (5.8) have sufficiently smooth particular solutions for arbitrary and sufficiently smooth right-hand terms. One can then assert that the particular integral of equation (5.1) can be constructed, with an arbitrary degree of accuracy, for an arbitrarily large k by the method proposed. If the level lines of the

change function of the right-hand side of equation (5.1) do not touch the characteristics of the operator L (as, in particular, is always the case when L is an elliptic operator), then the construction of a particular integral can be completed without solving any differential equations. Moreover, if in this case the right-hand side of equation (5.1) is bounded, then the particular integral will be of type $0(k^{-l})$. If the level lines of the change function of the right-hand side of equation (5.1) everywhere coincide with the characteristics of L, then the approximate particular integral is constructed by the successive solutions of first-order linear differential equations; hence, if the right-hand side of equation (5.1) is bounded, then the particular integral is of type $0(k^{-l+1})$. Thus, the case when the level lines of the function f coincide with the characteristics of L, is in a certain sense a case of resonance.

5. Let us suppose that the functions appearing in the right-hand side of (5.1) have the form

$$f(\alpha, \beta) = i\ (m\alpha + n\beta), \qquad \Psi(\alpha, \beta) = \text{const} \qquad (m, n = \text{const})$$

We then obtain an expression which is a high-order term of the complex Fourier series. Therefore, the question of how the particular integrals decrease with an increase of k is of interest in the investigation of the convergence of the solutions obtained by the established method of expanding the right-hand members of the differential equations into series.

6. We note that if the level lines of the function f do not coincide with the characteristics of L, that is, if the recurrence formulas are valid, then the approximate particular integral will be zero in every region where $\Psi = 0$. This means that if the function on the right-hand side of (5.1) is different from zero only in some subregion, and if in this subregion this function oscillates fast enough, and if the level lines of the change function do not coincide with the characteristics of the operator L, then one can construct such a particular integral of equation (5.1) as will be essentially different from zero only in the given subregion (it is of course assumed that the usual conditions of smoothness are satisfied by the parameters of the problem i.e. the function Ψ changes smoothly from its non-zero values to the value zero).

6. Example. We are given the following equation in polar coordinates

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + a(r, \theta)\right]\Phi = 0 \tag{6.1}$$

where a is a function on which for the time being no conditions are imposed. It is required to construct a solution of this equation lying within the circle $r \leqslant 1$, and satisfying the boundary conditions

$$\Phi = g_0 e^{ik\theta} \qquad \text{for } r = 1$$

(g_0 is a complex constant, k is a sufficiently large integer). In the case under consideration,

$$L_0 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

$$L_1 = 2\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{2}{r^2}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}$$

$$L_2 \equiv L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + a$$

The equation to determine f has the form

$$L_0 \equiv \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = 0$$

It is equivalent to two first-order linear differential equations

$$\frac{\partial f}{\partial r} + \frac{i}{r}\frac{\partial f}{\partial \theta} = 0, \qquad \frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta} = 0 \tag{6.2}$$

The following conditions must be imposed on the change function f

$$f = ig_0, \qquad R_e\left\{\frac{\partial f}{\partial r}\right\} > 0 \qquad \text{for } r=1 \tag{6.3}$$

The second of these conditions (the condition of damping) can be satisfied only by the solution of the first equation in (6.2). Hence, one can discard the second equation in (6.2). From equations (6.2) and (6.3) we obtain

$$f = \ln r + i\theta$$

With the aid of this equation, the operators L_1 and L_2 can be transformed to

$$L_1 = \frac{2}{r}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right), \qquad L_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + a$$

or

$$L_1 = \frac{4}{r^2}\frac{\partial}{\partial \bar{\rho}}, \qquad L_2 = \frac{4}{r^2}\frac{\partial^2}{\partial \rho \partial \bar{\rho}} + a \qquad \begin{pmatrix} \rho = \ln r + i\theta \\ \bar{\rho} = \ln r - i\theta \end{pmatrix}$$

We express the integral solution in the form

$$\Phi = e^{kf}(\Phi_0 + k^{-1}\Phi_1 + k^{-2}\Phi_2)$$

that is, we assume that the intensity function can be approximated by two terms of the expansion. The equations to determine Φ_0, Φ_1, Φ_2 will be

$$L_1(\Phi_0) = 0, \qquad L_1(\Phi_1) = -L_2(\Phi_0), \qquad k^{-1}L_2(\Phi_2) + L_1(\Phi_2) = -L_2(\Phi_1) \tag{6.4}$$

while the contour conditions for Φ_0, Φ_1, and Φ_2 are written as follows:

$$\Phi_0 = g_0, \qquad \Phi_1 = 0, \qquad \Phi_2 = 0 \qquad \text{at } r = 1 \tag{6.5}$$

The coefficients in the expansion of the change function are determined without difficulty by means of (6.4) and (6.5):

$$\Phi_0 = g_0, \qquad \Phi_1 = \int_{-i0}^{\bar{\rho}} \frac{ar^2}{4} g_0 d\rho$$

(a and r are considered as functions of ρ and $\bar{\rho}$).

To determine the remainder term we have the equations

$$\left[k^{-1}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + a\right) + \frac{2}{r}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)\right]\Phi_2 = g \tag{6.6}$$

where

$$g = g_0\left[\frac{1}{r^2}\frac{\partial}{\partial \rho}(ar^2) + a\int_{-i\theta}^{\bar{\rho}} \frac{ar^2}{4} d\bar{\rho}\right] \tag{6.7}$$

and the boundary conditions

$$\Phi_2 = 0 \qquad \text{at } r = 1 \tag{6.8}$$

The approximate solution (in which the remainder term is neglected) of the given problem can thus be written in the form

$$\Phi = e^{k(\ln r + i\theta)} g_0 \left(1 + k^{-1}\int_{-i0}^{\bar{\rho}} \frac{ar^2}{4} d\bar{\rho}\right)$$

If ar^2 is bounded as r goes to zero, then the solution is significant over the entire circle $r \leqslant 1$, for in this case it is not necessary to introduce a smoothing function.

The example considered belongs to the general class of problems treated by Vishik and Liusternik [2], and it can be solved by the method they proposed. By this method the form of the solution is the same as that obtained here, but Vishik and Liusternik replaced the coefficients of the equations in the first approximation by their contour values (since they were concerned with the construction of solutions "localized" near the contour γ). In this way the type of the change function is determined. In the work of Vishik and Liusternik, this function is always a linear function of the coordinate which determines the distance from the contour (in the case considered, it is a linear function of r). By the method proposed here, the change function is determined exactly at the first stage of the solution (in the case under consideration it is a logarithmic function of r), and in many cases this makes it possible to decrease the number of approximations required for obtaining a certain degree of accuracy (for a fixed k). In this connection it must be mentioned that each stage of the solution taken separately can be carried out more simply by the method of Vishik and Liusternik than by that proposed here.

7. The results obtained above admit of various generalizations.

They can of course be extended to the case of more than two independent variables. However, the method (described at the end of Section 1) for reducing the nonlinear equation determining f to l linear equations ceases to be applicable under this generalization.

The extension to the case of multiple characteristics is more difficult. In this case an integral (solution) of the form $\Phi = e^{kf}\,\Phi_*$ must be sought:

$$f = f_0 + \sum_{\lambda=0}^{\lambda=\zeta-\varkappa-1} k^{-(\varkappa+\lambda)/\zeta} f_{(\varkappa+\lambda)/\zeta}, \qquad \Phi_* = \sum_{u=\sigma+\tau/\zeta=0}^{u=R} k^{-u}\,\Phi_u$$

where κ, ζ are integers (not necessarily relatively prime to each other) and $\kappa \leqslant \zeta$; the summation in the formula for Φ_* is carried out over all u which are of the form $\sigma + \tau/\zeta$ (σ, τ are non-negative integers); the functions

$$f_0, \quad f_{\varkappa/\zeta}, \quad f_{(\varkappa+1)/\zeta}, \ldots, f_{(\zeta-1)/\zeta}, \qquad \Phi_0, \quad \Phi_{1/\zeta}, \ldots, \Phi_{R-1/\zeta} \tag{7.1}$$

do not depend on the parameter k.

The number κ/ζ can be so chosen as to obtain a recurrence process for the determination of the functions in (7.1). By this process the principal part of the change function will be determined the same way as the function f was determined earlier. The determination of the remaining terms of the sequence (7.1) can be reduced, in general, to the solution of first-order equations; however, cases can occur when it becomes necessary to solve equations of higher order in this connection.

If the multiplicity of some family of characteristics is p, then to this family there will correspond integrals (solutions) in which the principal part of the change function f_0 retains constant values on the curves of this family. In such a case, generally speaking, there will exist p processes to determine the remaining terms of the expansion of the change function. These processes will be distinguishable from each other either by the value of the number κ, or by the type of the differential equation from which the ratio $f_{\kappa/\zeta}$ is to be determined. Here also one can have exceptions. For a p-multiple characteristic in certain cases there may exist only $p - \pi$ of the processes described. In these cases, and these only, we will have the exceptions already mentioned, when it is necessary to solve equations of order higher than the first (namely of order π) to determine the coefficients of the expansion of the intensity function.

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